Base change and K-theory for $\mathrm{GL}(n, \mathbb{R})$

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Abstract

We investigate base change $\mathbb{C}/\mathbb{R}$ at the level of K-theory for the general linear group $\mathrm{GL}(n, \mathbb{R})$. In the course of this study, we compute in detail the C^*-algebra K-theory of this disconnected group. We investigate the interaction of base change with the Baum-Connes correspondence for $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$. This article is the archimedean companion of our previous article [9].

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1 Introduction

In the general theory of automorphic forms, an important role is played by *base change*. Base change has a global aspect and a local aspect [1]. In this article, we focus on the archimedean case of base change for the general linear group $\mathrm{GL}(n, \mathbb{R})$, and we investigate base change for this group at the level of K-theory.

For $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$ we have the Langlands classification and the associated L-parameters [7]. We recall that the domain of an L-parameter of $\mathrm{GL}(n, F)$ over an archimedean field F is the Weil group W_F. The Weil groups are given by

$$W_{\mathbb{C}} = \mathbb{C}^\times$$

and

$$W_{\mathbb{R}} = \mathbb{C}^\times \rtimes \mathbb{Z}/2\mathbb{Z}$$

where the generator of $\mathbb{Z}/2\mathbb{Z}$ sends a complex number z to its conjugate $\overline{z}$. Base change is defined by restriction of L-parameter from $W_{\mathbb{R}}$ to $W_{\mathbb{C}}$.

An L-parameter ϕ is *tempered* if $\phi(W_F)$ is bounded. Base change therefore determines a map of tempered duals.

In this article, we investigate the interaction of base change with the Baum-Connes correspondence for $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$.

Let F denote $\mathbb{R}$ or $\mathbb{C}$ and let $G = G(F) = \mathrm{GL}(n, F)$. Let $C_r^*(G)$ denote the reduced C^*-algebra of G. The Baum-Connes correspondence is a canonical isomorphism [8][5]

$$\mu_F : K_*^{G(F)}(\underline{E}G(F)) \to K_* C_r^*(G(F))$$

where $\underline{E}G(F)$ is a universal example for the action of $G(F)$.

The noncommutative space $C_r^*(G(F))$ is strongly Morita equivalent to the commutative C^*-algebra $C_0(\mathcal{A}_n^t(F))$ where $\mathcal{A}_n^t(F)$ denotes the tempered dual of $G(F)$, see [10, section 1.2][11]. As a consequence of this, we have

$$K_* C_r^*(G(F)) \cong K^* \mathcal{A}_n^t(F).$$

This leads to the following formulation of the Baum-Connes correspondence:

$$K_*^{G(F)}(\underline{E}G(F)) \cong K^* \mathcal{A}_n^t(F).$$

Base change $\mathbb{C}/\mathbb{R}$ determines a map

$$b_{\mathbb{C}/\mathbb{R}} : \mathcal{A}_n^t(\mathbb{R}) \to \mathcal{A}_n^t(\mathbb{C}).$$

This leads to the following diagram

$$\begin{array}{ccc}
K_*^{G(\mathbb{C})}(\underline{E}G(\mathbb{C})) & \xrightarrow{\mu_{\mathbb{C}}} & K^* \mathcal{A}_n^t(\mathbb{C})) \\
\big\downarrow & & \big\downarrow b^*_{\mathbb{C}/\mathbb{R}} \\
K_*^{G(\mathbb{R})}(\underline{E}G(\mathbb{R})) & \xrightarrow[\mu_{\mathbb{R}}]{} & K^* \mathcal{A}_n^t(\mathbb{R}).
\end{array}$$

where the left-hand vertical map is the unique map which makes the diagram commutative.

In section 2 we describe the tempered dual $\mathcal{A}_n^t(F)$ as a locally compact Hausdorff space.

In section 3 we recall base change for archimedean fields.

In section 4 we compute the K-theory for the reduced C^*-algebra of $\mathrm{GL}(n, \mathbb{R})$. We show that the K-theory depends on essentially one parameter q given by the maximum number of $2's$ in the partitions of n into $1's$ and $2's$. There are precisely $\lfloor \frac{n}{2} \rfloor + 1$ such partitions. If n is even then $q = n/2$ (Theorem 4.7) and if n is odd then $q = (n-1)/2$ (Theorem 4.8). The real reductive Lie group $\mathrm{GL}(n, \mathbb{R})$ is, of course, not connected. If n is even our formulas show that we always have non-trivial K^0 and K^1.

In section 5 we recall the K-theory for the reduced C^*-algebra of the complex reductive group $\mathrm{GL}(n, \mathbb{C})$, see [11].

In section 6 we compute the base change map $BC : \mathcal{A}_n^t(\mathbb{R}) \to \mathcal{A}_n^t(\mathbb{C})$ and prove that BC is a continuous proper map. At level of K-theory, base change is the zero map for $n > 1$ (Theorem 6.2).

In section 7, where we study the case $n = 1$, base change for K^1 creates a map

$$\mathcal{R}(\mathbb{T}) \longrightarrow \mathcal{R}(\mathbb{Z}/2\mathbb{Z})$$

where $\mathcal{R}(\mathbb{T})$ is the representation ring of the circle group $\mathbb{T}$ and $\mathcal{R}(\mathbb{Z}/2\mathbb{Z})$ is the representation ring of the group $\mathbb{Z}/2\mathbb{Z}$. This map sends the trivial character of $\mathbb{T}$ to $1 \oplus \varepsilon$, where ε is the nontrivial character of $\mathbb{Z}/2\mathbb{Z}$, and sends all the other characters of $\mathbb{T}$ to zero.

This map has an interpretation in terms of K-cycles. The K-cycle

$$(C_0(\mathbb{R}), L^2(\mathbb{R}), id/dx)$$

is equivariant with respect to $\mathbb{C}^\times$ and $\mathbb{R}^\times$, and therefore determines a class $\not\partial_{\mathbb{C}} \in K_1^{\mathbb{C}^\times}(\underline{E}\mathbb{C}^\times)$ and a class $\not\partial_{\mathbb{R}} \in K_1^{\mathbb{R}^\times}(\underline{E}\mathbb{R}^\times)$. On the left-hand-side of the Baum-Connes correspondence, base change in dimension 1 admits the following description:

$$\not\partial_{\mathbb{C}} \mapsto (\not\partial_{\mathbb{R}}, \not\partial_{\mathbb{R}})$$

This extends the results of [9] to archimedean fields.

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2 On the tempered dual of $\mathrm{GL}(n)$

Let $F = \mathbb{R}$. In order to compute the K-theory of the reduced C^*-algebra of $\mathrm{GL}(n, F)$ we need to parametrize the tempered dual $\mathcal{A}_n^t(F)$ of $GL(n, F)$.

Let M be a standard Levi subgroup of $\mathrm{GL}(n, F)$, i.e. a block-diagonal subgroup. Let ${}^0M$ be the subgroup of M such that the determinant of each block-diagonal is ± 1. Denote by $X(M) = \widehat{M/{}^0M}$ the group of *unramified characters* of M, consisting of those characters which are trivial on ${}^0M$.

Let $W(M) = N(M)/M$ denote the Weyl group of M. $W(M)$ acts on the discrete series $E_2({}^0M)$ of ${}^0M$ by permutations.

Now, choose one element $\sigma \in E_2({}^0M)$ for each $W(M)$-orbit. The *isotropy subgroup* of $W(M)$ is defined to be

$$W_\sigma(M) = \{\omega \in W(M) : \omega.\sigma = \sigma\}.$$

Form the disjoint union

$$\bigsqcup_{(M,\sigma)} X(M)/W_\sigma(M) = \bigsqcup_{M} \bigsqcup_{\sigma \in E_2(^0M)} X(M)/W_\sigma(M). \tag{1}$$

The disjoint union has the structure of a locally compact, Hausdorff space and is called the *Harish-Chandra parameter space*. The parametrization of the tempered dual $\mathcal{A}_n^t(\mathbb{R})$ is due to Harish-Chandra, see [6].

Proposition 2.1 (Harish-Chandra)**.** *[6] There exists a bijection*

$$\begin{array}{ccc} \bigsqcup_{(M,\sigma)} X(M)/W_\sigma(M) & \longrightarrow & \mathcal{A}_n^t(\mathbb{R}) \\ \chi^\sigma & \mapsto & i_{GL(n),MN}(\chi^\sigma \otimes 1), \end{array}$$

where $\chi^\sigma(x) := \chi(x)\sigma(x)$ *for all* $x \in M$*.*

In view of the above bijection, we will denote the Harish-Chandra parameter space by $\mathcal{A}_n^t(\mathbb{R})$.

We will see now the particular features of the archimedean case, starting with $GL(n,\mathbb{R})$. Since the discrete series of $GL(n,\mathbb{R})$ is empty for $n \geq 3$, we only need to consider partitions of n into 1's and 2's. This allows us to to decompose n as $n = 2q + r$, where q is the number of 2's and r is the number of 1's in the partition. To this decomposition we associate the partition

$$n = (\underbrace{2, ..., 2}_{q}, \underbrace{1, ..., 1}_{r}),$$

which corresponds to the Levi subgroup

$$M \cong \underbrace{GL(2,\mathbb{R}) \times ... \times GL(2,\mathbb{R})}_{q} \times \underbrace{GL(1,\mathbb{R}) \times ... \times GL(1,\mathbb{R})}_{r}.$$

Varying q and r we determine a representative in each equivalence class of Levi subgroups. The subgroup ^0M of M is given by

$$^0M \cong \underbrace{SL^{\pm}(2,\mathbb{R}) \times ... \times SL^{\pm}(2,\mathbb{R})}_{q} \times \underbrace{SL^{\pm}(1,\mathbb{R}) \times ... \times SL^{\pm}(1,\mathbb{R})}_{r},$$

where

$$SL^{\pm}(m,\mathbb{R}) = \{g \in GL(m,\mathbb{R}) : |det(g)| = 1\}$$

is the *unimodular subgroup* of $GL(m,\mathbb{R})$. In particular, $SL^{\pm}(1,\mathbb{R}) = \{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$.

The representations in the discrete series of $GL(2,\mathbb{R})$, denoted $\mathcal{D}_\ell$ for $\ell \in \mathbb{N}$ ($\ell \geq 1$) are induced from $SL(2,\mathbb{R})$ [7, p.399]:

$$\mathcal{D}_\ell = ind_{SL^\pm(2,\mathbb{R}),SL(2,\mathbb{R})}(\mathcal{D}_\ell^\pm),$$

where $\mathcal{D}_\ell^\pm$ acts in the space

$$\{f : \mathcal{H} \to \mathbb{C} | f \text{ analytic }, \|f\|^2 = \int\int |f(z)|^2 y^{\ell-1}) dxdy < \infty\}.$$

Here, $\mathcal{H}$ denotes the Poincaré upper half plane. The action of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by

$$\mathcal{D}_\ell^\pm(g)(f(z)) = (bz+d)^{-(\ell+1)} f(\frac{az+c}{bz+d}).$$

More generally, an element σ from the discrete series $E_2({}^0M)$ is given by

$$\sigma{=}i_{G,MN}(\mathcal{D}_{\ell_1}^\pm \otimes ... \otimes \mathcal{D}_{\ell_q}^\pm \otimes \tau_1 \otimes ... \otimes \tau_r \otimes 1), \tag{2}$$

where $\mathcal{D}_{\ell_i}^\pm$ ($\ell_i \geq 1$) are the discrete series representations of $SL^\pm(2,\mathbb{R})$ and τ_j is a representation of $SL^\pm(1,\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$, i.e. $id = (x \mapsto x)$ or $sgn = (x \mapsto \frac{x}{|x|})$.

Finally we will compute the unramified characters $X(M)$, where M is the Levi subgroup associated to the partition $n = 2q + r$.

Let $x \in GL(2,\mathbb{R})$. Any character of $GL(2,\mathbb{R})$ is given by

$$\chi(det(x)) = (sgn(det(x)))^\varepsilon |det(x)|^{it}$$

($\varepsilon = 0, 1, t \in \mathbb{R}$) and it is unramified provided that

$$\chi(det(g)) = \chi(\pm 1) = (\pm 1)^\varepsilon = 1, \text{ for all } g \in SL^\pm(2,\mathbb{R}).$$

This implies $\varepsilon = 0$ and any unramified character of $GL(2,\mathbb{R})$ has the form

$$\chi(x) = |det(x)|^{it}, \text{ for some } t \in \mathbb{R}. \tag{3}$$

Similarly, any unramified character of $GL(1,\mathbb{R}) = \mathbb{R}^\times$ has the form

$$\xi(x) = |x|^{it}, \text{ for some } t \in \mathbb{R}. \tag{4}$$

Given a block diagonal matrix $diag(g_1, ..., g_q, \omega_1, ..., \omega_r) \in M$, where $g_i \in GL(2,\mathbb{R})$ and $\omega_j \in GL(1,\mathbb{R})$, we conclude from (3) and (4) that any unramified character $\chi \in X(M)$ is given by

$$\begin{aligned} &\chi(diag(g_1, ..., g_q, \omega_1, ..., \omega_r)) = \\ &= |det(g_1)|^{it_1} \times ... \times |det(g_q)|^{it_q} \times |\omega_1|^{it_{q+1}} \times ... \times |\omega_r|^{it_{q+r}}, \end{aligned}$$

for some $(t_1, ..., t_{q+r}) \in \mathbb{R}^{q+r}$. We can denote such element $\chi \in X(M)$ by $\chi_{(t_1,...,t_{q+r})}$. We have the following result.

Proposition 2.2. *Let M be a Levi subgroup of $GL(n,\mathbb{R})$, associated to the partition $n = 2q + r$. Then, there is a bijection*

$$X(M) \to \mathbb{R}^{q+r}\ ,\ \chi_{(t_1,...,t_{q+r})} \mapsto (t_1, ..., t_{q+r}).$$

Let us consider now $GL(n,\mathbb{C})$. The tempered dual of $GL(n,\mathbb{C})$ comprises the *unitary principal series* in accordance with Harish-Chandra [10, p. 277]. The corresponding Levi subgroup is a maximal torus $T \cong (\mathbb{C}^\times)^n$. It follows that ${}^0T \cong \mathbb{T}^n$ the compact n-torus.

The principal series representations are given by

$$\pi_{\ell,it} = i_{G,TU}(\sigma \otimes 1), \tag{5}$$

where σ=$\sigma_1 \otimes ... \otimes \sigma_n$ and $\sigma_j(z) = (\frac{z}{|z|})^{\ell_j}|z|^{it_j}$ ($\ell_j \in \mathbb{Z}$ and $t_j \in \mathbb{R}$).

An unramified character is given by

$$\chi\begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix} = |z_1|^{it_1} \times ... \times |z_n|^{it_n}$$

and we can represent χ as $\chi_{(t_1,...,t_n)}$. Therefore, we have the following result.

Proposition 2.3. *Denote by T the standard maximal torus in $GL(n,\mathbb{C})$. There is a bijection*

$$X(T) \to \mathbb{R}^n\ ,\ \chi_{(t_1,...,t_n)} \mapsto (t_1, ..., t_n).$$

3 Base change for archimedean fields

The Weil group attached to a local field F will be denoted W_F as in [13]. We may state the base change problem for archimedean fields in the following way. Consider the archimedean base change $\mathbb{C}/\mathbb{R}$. We have $W_\mathbb{C} \subset W_\mathbb{R}$ and there is a natural map

$$Res^{W_\mathbb{R}}_{W_\mathbb{C}} : \mathcal{G}_n(\mathbb{R}) \longrightarrow \mathcal{G}_n(\mathbb{C}) \tag{6}$$

called *restriction*. By the local Langlands correspondence for archimedean fields [3, Theorem 3.1, p.236][7], there is a base change map

$$BC : \mathcal{A}_n(\mathbb{R}) \longrightarrow \mathcal{A}_n(\mathbb{C}) \tag{7}$$

such that the following diagram commutes

$$\begin{tikzcd} \mathcal{A}_n(\mathbb{R}) \arrow[r, "BC"] & \mathcal{A}_n(\mathbb{C}) \\ \mathcal{G}_n(\mathbb{R}) \arrow[u, "{}_{\mathbb{R}}\mathcal{L}_n"] \arrow[r, "Res^{W_{\mathbb{R}}}_{W_{\mathbb{C}}}"'] & \mathcal{G}_n(\mathbb{C}) \arrow[u, "{}_{\mathbb{C}}\mathcal{L}_n"'] \end{tikzcd}$$

Arthur and Clozel's book [1] gives a full treatment of base change for $GL(n)$. The case of archimedean base change can be captured in an elegant formula [1, p. 71]. We briefly review these results.

Given a partition $n = 2q + r$ let χ_i $(i = 1, ..., q)$ be a ramified character of $\mathbb{C}^\times$ and let ξ_j $(j = 1, ..., r)$ be a ramified character of $\mathbb{R}^\times$. Since the χ_i's are ramified, $\chi_i(z) \neq \chi_i^\sigma(z) = \chi_i(\overline{z})$. By Langlands classification [7], each χ_i defines a discrete series representation $\pi(\chi_i)$ of $GL(2, \mathbb{R})$, with $\pi(\chi_i) = \pi(\chi_i^\sigma)$. Denote by $\pi(\chi_1, ..., \chi_q, \xi_1, ..., \xi_r)$ the *generalized principal series representation* of $GL(n, \mathbb{R})$

$$\pi(\chi_1, ..., \chi_q, \xi_1, ..., \xi_r) = i_{GL(n,\mathbb{R}),MN}(\pi(\chi_1)\otimes ...\otimes\pi(\chi_q)\otimes\xi_1\otimes ...\otimes\xi_r\otimes 1). \quad (8)$$

The base change map for the general principal series representation is given by induction from the Borel subgroup $B(\mathbb{C})$ [1, p. 71]:

$$BC(\pi) = \Pi(\chi_1, ..., \chi_q, \xi_1, ..., \xi_r) = i_{GL(n,\mathbb{C}),B(\mathbb{C})}(\chi_1, \chi_1^\sigma, ..., \chi_q, \chi_q^\sigma, \xi_1\circ N, ..., \xi_r\circ N), \quad (9)$$

where $N = N_{\mathbb{C}/\mathbb{R}} : \mathbb{C}^\times \longrightarrow \mathbb{R}^\times$ is the norm map defined by $z \mapsto z\overline{z}$.

We illustrate the base change map with two simple examples.

Example 3.1. *For $n = 1$, base change is simply composition with the norm map*

$$BC : \mathcal{A}_1^t(\mathbb{R}) \to \mathcal{A}_1^t(\mathbb{C}) \ , \ BC(\chi) = \chi \circ N.$$

Example 3.2. *For $n = 2$, there are two different kinds of representations, one for each partition of 2. According to (8), $\pi(\chi)$ corresponds to the partition $2 = 2 + 0$ and $\pi(\xi_1, \xi_2)$ corresponds to the partition $2 = 1 + 1$. Then the base change map is given, respectively, by*

$$BC(\pi(\chi)) = i_{GL(2,\mathbb{C}),B(\mathbb{C})}(\chi, \chi^\sigma),$$

and

$$BC(\pi(\xi_1, \xi_2)) = i_{GL(2,\mathbb{C}),B(\mathbb{C})}(\xi_1 \circ N, \xi_2 \circ N).$$

4 K-theory for $GL(n,\mathbb{R})$

Using the Harish-Chandra parametrization of the tempered dual of $GL(n)$ (recall that the Harish-Chandra parameter space is a locally compact, Hausdorff topological space) we can compute the K-theory of the reduced C^*-algebra $C_r^*GL(n,\mathbb{R})$.

We have

$$\begin{aligned} K_j(C_r^*GL(n,\mathbb{R})) &= K^j(\bigsqcup_{(M,\sigma)} X(M)/W_\sigma(M)) \\ &= \bigoplus_{(M,\sigma)} K^j(X(M)/W_\sigma(M)) \\ &= \bigoplus_{(M,\sigma)} K^j(\mathbb{R}^{n_M}/W_\sigma(M)), \end{aligned} \tag{10}$$

where $n_M = q + r$ if M is a representative of the equivalence class of Levi subgroup associated to the partition $n = 2q+r$. Hence the K-theory depends on n and on each Levi subgroup.

To compute (10) we have to consider the following orbit spaces:

(i) $\mathbb{R}^n$, in which case $W_\sigma(M)$ is the trivial subgroup of the Weil group $W(M)$;

(ii) $\mathbb{R}^n/S_n$, where $W_\sigma(M) = W(M)$ (this is one of the possibilities for the partition of n into 1's);

(iii) $\mathbb{R}^n/(S_{n_1} \times ... \times S_{n_k})$, where $W_\sigma(M) = S_{n_1} \times ... \times S_{n_k} \subset W(M)$ (see the examples below).

Definition 4.1. *An orbit space as indicated in* (*ii*) *and* (*iii*) *is called a closed cone.*

The K-theory for $\mathbb{R}^n$ may be summarized as follows

$$K^j(\mathbb{R}^n) = \begin{cases} \mathbb{Z}, n = j(mod 2) \\ 0, otherwise. \end{cases}$$

The next results show that the K-theory of a closed cone vanishes.

Lemma 4.2. $K^j(\mathbb{R}^n/S_n) = 0, j = 0,1.$

Proof. We need the following definition. A point $(a_1, ..., a_n) \in \mathbb{R}^n$ is called normalized if $a_j \leq a_{j+1}$, for $j = 1, 2, ..., n-1$. Therefore, in each orbit there is exactly one normalized point and $\mathbb{R}^n/S_n$ is homeomorphic to the subset of $\mathbb{R}^n$ consisting of all normalized points of $\mathbb{R}^n$. We denote the set of all normalized points of $\mathbb{R}^n$ by $N(\mathbb{R}^n)$.

In the case of $n = 2$, let (a_1, a_2) be a normalized point of $\mathbb{R}^2$. Then, there is a unique $t \in [1, +\infty[$ such that $a_2 = ta_1$ and the map

$$\mathbb{R} \times [1, +\infty[\to N(\mathbb{R}^2), (a, t) \mapsto (a, ta)$$

is a homeomorphism.

If $n > 2$ then the map

$$N(\mathbb{R}^{n-1}) \times [1, +\infty[\to N(\mathbb{R}^n), (a_1, ..., a_{n-1}, t) \mapsto (a_1, ..., a_{n-1}, ta_n)$$

is a homeomorphism. Since $[1, +\infty[$ kills both the K-theory groups K^0 and K^1, the result follows by applying Künneth formula. $\square$

The symmetric group S_n acts on $\mathbb{R}^n$ by permuting the components. This induces an action of any subgroup $S_{n_1} \times ... \times S_{n_k}$ of S_n on $\mathbb{R}^n$. Write

$$\mathbb{R}^n \cong \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times ... \times \mathbb{R}^{n_k} \times \mathbb{R}^{n-n_1-...-n_k}.$$

If $n = n_1 + ... + n_k$ then we simply have $\mathbb{R}^n \cong \mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_k}$.

$S_{n_1} \times ... \times S_{n_k}$ acts on $\mathbb{R}^n$ as follows.
S_{n_1} permutes the components of $\mathbb{R}^{n_1}$ leaving the remaining fixed;
S_{n_2} permutes the components of $\mathbb{R}^{n_2}$ leaving the remaining fixed;
and so on. If $n > n_1 + ... + n_k$ the components of $\mathbb{R}^{n-n_1-...-n_k}$ remain fixed. This can be interpreted, of course, as the action of the trivial subgroup. As a consequence, one identify the orbit spaces

$$\mathbb{R}^n/(S_{n_1} \times ... \times S_{n_k}) \cong \mathbb{R}^{n_1}/S_{n_1} \times ... \times \mathbb{R}^{n_k}/S_{n_k} \times \mathbb{R}^{n-n_1-...-n_k}$$

Lemma 4.3. *$K^j(\mathbb{R}^n/(S_{n_1} \times ... \times S_{n_k}) = 0, j = 0, 1,$ where $S_{n_1} \times ... \times S_{n_k} \subset S_n$.*

Proof. It suffices to prove for $\mathbb{R}^n/(S_{n_1} \times S_{n_2})$. The general case follows by induction on k.

Now, $\mathbb{R}^n/(S_{n_1} \times S_{n_2}) \cong \mathbb{R}^{n_1}/S_{n_1} \times \mathbb{R}^{n-n_1}/S_{n_2}$. Applying the Künneth formula and Lemma 4.2, the result follows. $\square$

We give now some examples by computing $K_jC_r^*GL(n, \mathbb{R})$ for small n.

Example 4.4. *We start with the case of $GL(1, \mathbb{R})$. We have:*

$$M = \mathbb{R}^\times \ , \ {}^0M = \mathbb{Z}/2\mathbb{Z} \ , \ W(M) = 1 \text{ and } X(M) = \mathbb{R}.$$

Hence,

$$\mathcal{A}_1^t(\mathbb{R}) \cong \bigsqcup_{\sigma \in \widehat{(\mathbb{Z}/2\mathbb{Z})}} \mathbb{R}/1 = \mathbb{R} \sqcup \mathbb{R}, \tag{11}$$

and the K-theory is given by

$$K_jC_r^*GL(1, \mathbb{R}) \cong K^j(\mathcal{A}_1^t(\mathbb{R})) = K^j(\mathbb{R} \sqcup \mathbb{R}) = K^j(\mathbb{R}) \oplus K^j(\mathbb{R}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & , j = 1 \\ 0 & , j = 0. \end{cases}$$

Example 4.5. *For $GL(2,\mathbb{R})$ we have two partitions of $n = 2$ and the following data*

Partition	M	${}^0M$	$W(M)$	$X(M)$	$\sigma \in E_2({}^0M)$
2+0	$GL(2,\mathbb{R})$	$SL^{\pm}(2,\mathbb{R})$	1	$\mathbb{R}$	$\sigma = i_{G,P}(\mathcal{D}_\ell^+), \ell \in \mathbb{N}$
1+1	$(\mathbb{R}^\times)^2$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{R}^2$	$\sigma = i_{G,P}(id \otimes sgn)$

Then the tempered dual is parameterized as follows

$$\mathcal{A}_2^t(\mathbb{R}) \cong \bigsqcup_{(M,\sigma)} X(M)/W_\sigma(M) = (\bigsqcup_{\ell\in\mathbb{N}} \mathbb{R}) \sqcup (\mathbb{R}^2/S_2) \sqcup (\mathbb{R}^2/S_2) \sqcup \mathbb{R}^2,$$

and the K-theory groups are given by

$$K_jC_r^*GL(2,\mathbb{R}) \cong K^j(\mathcal{A}_2^t(\mathbb{R})) \cong (\bigoplus_{\ell\in\mathbb{N}} K^j(\mathbb{R})) \oplus K^j(\mathbb{R}^2) = \begin{cases} \bigoplus_{\ell\in\mathbb{N}} \mathbb{Z} & , j = 1 \\ \mathbb{Z} & , j = 0. \end{cases}$$

Example 4.6. *For $GL(3,\mathbb{R})$ there are two partitions for $n = 3$, to which correspond the following data*

Partition	M	${}^0M$	$W(M)$	$X(M)$
2+1	$GL(2,\mathbb{R}) \times \mathbb{R}^\times$	$SL^{\pm}(2,\mathbb{R}) \times (\mathbb{Z}/2\mathbb{Z})$	1	$\mathbb{R}^2$
1+1+1	$(\mathbb{R}^\times)^3$	$(\mathbb{Z}/2\mathbb{Z})^3$	S_3	$\mathbb{R}^3$

For the partition $3 = 2 + 1$, an element $\sigma \in E_2({}^0M)$ is given by

$$\sigma = i_{G,P}(\mathcal{D}_\ell^+ \otimes \tau) \ , \ \ell \in \mathbb{N} \text{ and } \tau \in (\widehat{\mathbb{Z}/2\mathbb{Z}}).$$

For the partition $3 = 1 + 1 + 1$, an element $\sigma \in E_2({}^0M)$ is given by

$$\sigma = i_{G,P}(\bigotimes_{i=1}^{3} \tau_i) \ , \ \tau_i \in (\widehat{\mathbb{Z}/2\mathbb{Z}}).$$

The tempered dual is parameterized as follows

$$\mathcal{A}_3^t(\mathbb{R}) \cong \bigsqcup_{(M,\sigma)} X(M)/W_\sigma(M) = \bigsqcup_{\mathbb{N}\times(\mathbb{Z}/2\mathbb{Z})} (\mathbb{R}^2/1) \bigsqcup_{(\mathbb{Z}/2\mathbb{Z})^3} (\mathbb{R}^3/S_3).$$

The K-theory groups are given by

$$K_jC_r^*GL(3,\mathbb{R}) \cong K^j(\mathcal{A}_3^t(\mathbb{R})) \cong \bigoplus_{\mathbb{N}\times(\mathbb{Z}/2\mathbb{Z})} K^j(\mathbb{R}^2) \oplus 0 = \begin{cases} \bigoplus_{\mathbb{N}\times(\mathbb{Z}/2\mathbb{Z})} \mathbb{Z} & , j = 0 \\ 0 & , j = 1. \end{cases}$$

The general case of $GL(n,\mathbb{R})$ will now be considered. It can be split in two cases: n even and n odd.

- $n = 2q$ even

Suppose n is even. For every partition $n = 2q + r$, either $W_\sigma(M) = 1$ or $W_\sigma(M) \neq 1$. If $W_\sigma(M) \neq 1$ then $\mathbb{R}^{n_M}/W_\sigma(M)$ is a cone and the K-groups K^0 and K^1 both vanish. This happens precisely when $r > 2$ and therefore we have only two partitions, corresponding to the choices of $r = 0$ and $r = 2$, which contribute to the K-theory with non-zero K-groups

Partition	M	${}^0M$	$W(M)$
$2q$	$GL(2,\mathbb{R})^q$	$SL^{\pm}(2,\mathbb{R})^q$	S_q
$2(q-1)+2$	$GL(2,\mathbb{R})^{q-1} \times (\mathbb{R}^\times)^2$	$SL^{\pm}(2,\mathbb{R})^{q-1} \times (\mathbb{Z}/2\mathbb{Z})^2$	$S_{q-1} \times (\mathbb{Z}/2\mathbb{Z})$

We also have $X(M) \cong \mathbb{R}^q$ for $n = 2q$, and $X(M) \cong \mathbb{R}^{q+1}$, for $n = 2(q-1)+2$.

For the partition $n = 2q$ $(r = 0)$, an element $\sigma \in E_2({}^0M)$ is given by

$$\sigma = i_{G,P}(\mathcal{D}^+_{\ell_1} \otimes ... \otimes \mathcal{D}^+_{\ell_q}) \ , (\ell_1, ...\ell_q) \in \mathbb{N}^q \text{ and } \ell_i \neq \ell_j \text{ if } i \neq j.$$

For the partition $n = 2(q-1)+2$ $(r = 2)$, an element $\sigma \in E_2({}^0M)$ is given by

$$\sigma = i_{G,P}(\mathcal{D}^+_{\ell_1} \otimes ... \otimes \mathcal{D}^+_{\ell_{q-1}} \otimes id \otimes sgn) \ , (\ell_1, ...\ell_{q-1}) \in \mathbb{N}^{q-1} \text{ and } \ell_i \neq \ell_j \text{ if } i \neq j.$$

Therefore, the tempered dual has the following form

$$\mathcal{A}^t_n(\mathbb{R}) = \mathcal{A}^t_{2q}(\mathbb{R}) = (\bigsqcup_{\ell \in \mathbb{N}^q} \mathbb{R}^q) \sqcup (\bigsqcup_{\ell' \in \mathbb{N}^{q-1}} \mathbb{R}^{q+1}) \sqcup \mathcal{C}$$

where $\mathcal{C}$ is a disjoint union of closed cones as in Definition 4.1.

Theorem 4.7. *Suppose $n = 2q$ is even. Then the K-groups are*

$$K_j C^*_r GL(n,\mathbb{R}) \cong \begin{cases} \bigoplus_{\ell \in \mathbb{N}^q} \mathbb{Z} & , j \equiv q (mod 2) \\ \bigoplus_{\ell \in \mathbb{N}^{q-1}} \mathbb{Z} & , otherwise. \end{cases}$$

If $q = 1$ then the direct sum $\bigoplus_{\ell \in \mathbb{N}^{q-1}} \mathbb{Z}$ will denote a single copy of $\mathbb{Z}$.

- $n = 2q + 1$ odd

If n is odd only one partition contributes to the K-theory of $GL(n,\mathbb{R})$ with non-zero K-groups:

Partition	M	${}^0M$	$W(M)$	$X(M)$
$2q+1$	$GL(2,\mathbb{R})^{q+1}\times\mathbb{R}^\times$	$SL^\pm(2,\mathbb{R})^q\times(\mathbb{Z}/2\mathbb{Z})$	S_q	$\mathbb{R}^{q+1}$

An element $\sigma\in E_2({}^0M)$ is given by

$\sigma = i_{G,P}(\mathcal{D}^+_{\ell_1}\otimes ...\otimes\mathcal{D}^+_{\ell_q}\otimes\tau)$, $(\ell_1,...\ell_q,\tau)\in\mathbb{N}^q\times(\mathbb{Z}/2\mathbb{Z})$ and $\ell_i\neq\ell_j$ if $i\neq j$.

The tempered dual is given by

$$\mathcal{A}^t_n(\mathbb{R}) = \mathcal{A}^t_{2q+1}(\mathbb{R}) = (\bigsqcup_{\ell\in(\mathbb{N}^q\times(\mathbb{Z}/2\mathbb{Z}))}\mathbb{R}^{q+1})\sqcup\mathcal{C}$$

where $\mathcal{C}$ is a disjoint union of closed cones as in Definition 4.1.

Theorem 4.8. *Suppose $n=2q+1$ is odd. Then the K-groups are*

$$K_jC^*_rGL(n,\mathbb{R})\cong\begin{cases}\bigoplus_{\ell\in\mathbb{N}^q\times(\mathbb{Z}/2\mathbb{Z})}\mathbb{Z} & ,j\equiv q+1(mod2)\\ 0 & ,\textit{otherwise.}\end{cases}$$

Here, we use the following convention: if $q=0$ then the direct sum is $\bigoplus_{\mathbb{Z}/2\mathbb{Z}}\mathbb{Z}\cong\mathbb{Z}\oplus\mathbb{Z}$.

We conclude that the K-theory of $C^*_rGL(n,\mathbb{R})$ depends on essentially one parameter q given by the maximum number of $2's$ in the partitions of n into $1's$ and $2's$. If n is even then $q=\frac{n}{2}$ and if n is odd then $q=\frac{n-1}{2}$.

5 K-theory for $\mathrm{GL}(n,\mathbb{C})$

Let T° be the maximal compact subgroup of the maximal compact torus T of $\mathrm{GL}(n,\mathbb{C})$. Let σ be a unitary character of T°. We note that $W=W(T)$, $W_\sigma=W_\sigma(T)$. If $W_\sigma=1$ then we say that the orbit $W\cdot\sigma$ is *generic*.

Theorem 5.1. *The K-theory of $C^*_r\mathrm{GL}(n,\mathbb{C})$ admits the following description. If $n=j \mod 2$ then K_j is free abelian on countably many generators, one for each generic W- orbit in the unitary dual of T°, and $K_{j+1}=0$.*

Proof. We have a homeomorphism of locally compact Hausdorff spaces:

$$\mathcal{A}^t_n(\mathbb{C})\cong\bigsqcup X(T)/W_\sigma(T)$$

by the Harish-Chandra Plancherel Theorem for complex reductive groups [6], and the identification of the Jacobson topology on the left-hand-side with the natural topology on the right-hand-side, as in [11] . The result now follows from Lemma 4.3. ∎

6 The base change map

In this section we define base change as a map of topological spaces and study the induced K-theory map.

Proposition 6.1. *The base change map $BC : \mathcal{A}_n^t(\mathbb{R}) \to \mathcal{A}_n^t(\mathbb{C})$ is a continuous proper map.*

Proof. First, we consider the case $n = 1$.
As we have seen in Example 3.1, base change for $GL(1)$ is the map given by $BC(\chi) = \chi \circ N$, for all characters $\chi \in \mathcal{A}_1^t(\mathbb{R})$, where $N : \mathbb{C}^\times \to \mathbb{R}^\times$ is the norm map.

Let $z \in \mathbb{C}^\times$. We have

$$BC(\chi)(z) = \chi(|z|^2) = |z|^{2it}. \tag{12}$$

A generic element from $\mathcal{A}_1^t(\mathbb{C})$ has the form

$$\mu(z) = (\frac{z}{|z|})^\ell |z|^{it}, \tag{13}$$

where $\ell \in \mathbb{Z}$ and $t \in S^1$, as stated before. Viewing the Pontryagin duals $\mathcal{A}_1^t(\mathbb{R})$ and $\mathcal{A}_1^t(\mathbb{C})$ as topological spaces by forgetting the group structure, and comparing (12) and (13), the base change map can be defined as the following continuous map

$$\begin{array}{ccc} \varphi : \mathcal{A}_1^t(\mathbb{R}) \cong \mathbb{R} \times (\mathbb{Z}/2\mathbb{Z}) & \longrightarrow & \mathcal{A}_1^t(\mathbb{C}) \cong \mathbb{R} \times \mathbb{Z} \\ \chi = (t, \varepsilon) & \mapsto & (2t, 0) \end{array}$$

A compact subset of $\mathbb{R} \times \mathbb{Z}$ in the connected component $\{\ell\}$ of $\mathbb{Z}$ has the form $K \times \{\ell\} \subset \mathbb{R} \times \mathbb{Z}$, where $K \subset \mathbb{R}$ is compact. We have

$$\varphi^{-1}(K \times \{\ell\}) = \begin{cases} \emptyset & , \text{if } \ell \neq 0 \\ \frac{1}{2}K \times \{\varepsilon\} & , \text{if } \ell = 0, \end{cases}$$

where $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$. Therefore $\varphi^{-1}(K \times \{\ell\})$ is compact and φ is proper.

The Case $n > 1$
Base change determines a map $BC : \mathcal{A}_n^t(\mathbb{R}) \to \mathcal{A}_n^t(\mathbb{C})$ of topological spaces. Let $X = X(M)/W_\sigma(M)$ be a connected component of $\mathcal{A}_n^t(\mathbb{R})$. Then, X is mapped under BC into a connected component $Y = Y(T)/W_{\sigma'}(T)$ of $\mathcal{A}_n^t(\mathbb{C})$. Given a generalized principal series representation

$$\pi(\chi_1, ..., \chi_q, \xi_1, ..., \xi_r)$$

where the χ_i's are ramified characters of $\mathbb{C}^\times$ and the ξ's are ramified characters of $\mathbb{R}^\times$, then

$$BC(\pi) = i_{G,B}(\chi_1, \chi_1^\tau, ..., \chi_q, \chi_q^\tau, \xi_1 \circ N, ..., \xi_r \circ N).$$

Here, $N = N_{\mathbb{C}/\mathbb{R}}$ is the norm map and τ is the generator of $Gal(\mathbb{C}/\mathbb{R})$.

We associate to π the usual parameters uniquely defined for each character χ and ξ. For simplicity, we write the set of parameters as a $(q+r)$-uple:

$$(t, t') = (t_1, ..., t_q, t'_1, ..., t'_r) \in \mathbb{R}^{q+r} \cong X(M).$$

Now, if $\pi(\chi_1, ..., \chi_q, \xi_1, ..., \xi_r)$ lies in the connected component defined by the fixed parameters $(\ell, \varepsilon) \in \mathbb{Z}^q \times (\mathbb{Z}/2\mathbb{Z})^r$, then

$$(t, t') \in X(M) \mapsto (t, t, 2t') \in Y(T)$$

is a continuous proper map.

It follows that

$$BC : X(M)/W_\sigma(M) \to Y(T)/W_{\sigma'}(T)$$

is continuous and proper since the orbit spaces are endowed with the quotient topology. □

Theorem 6.2. *The functorial map induced by base change*

$$K_j(C_r^* GL(n, \mathbb{C})) \stackrel{K_j(BC)}{\longrightarrow} K_j(C_r^* GL(n, \mathbb{R}))$$

is zero for $n > 1$.

Proof. The case $n > 2$

We start with the case $n > 2$. Let $n = 2q + r$ be a partition and M the associated Levi subgroup of $GL(n, \mathbb{R})$. Denote by $X_\mathbb{R}(M)$ the unramified characters of M. As we have seen, $X_\mathbb{R}(M)$ is parametrized by $\mathbb{R}^{q+r}$. On the other hand, the only Levi subgroup of $GL(n, \mathbb{C})$ for $n = 2q+r$ is the diagonal subgroup $X_\mathbb{C}(M) = (\mathbb{C}^\times)^{2q+r}$.

If $q = 0$ then $r = n$ and both $X_\mathbb{R}(M)$ and $X_\mathbb{C}(M)$ are parametrized by $\mathbb{R}^n$. But then in the real case an element $\sigma \in E_2(^0M)$ is given by

$$\sigma = i_{GL(n,\mathbb{R}),P}(\chi_1 \otimes ... \otimes \chi_n),$$

with $\chi_i \in \widehat{\mathbb{Z}/2\mathbb{Z}}$. Since $n \geq 3$ there is always repetition of the χ_i's. It follows that the isotropy subgroups $W_\sigma(M)$ are all nontrivial and the quotient spaces $\mathbb{R}^n/W_\sigma$ are closed cones. Therefore, the K-theory groups vanish.

If $q \neq 0$, then $X_{\mathbb{R}}(M)$ is parametrized by $\mathbb{R}^{q+r}$ and $X_{\mathbb{C}}(M)$ is parametrized by $\mathbb{R}^{2q+r}$ (see Propositions 2.2 and 2.3).

Base change creates a map

$$\mathbb{R}^{q+r} \longrightarrow \mathbb{R}^{2q+r}.$$

Composing with the stereographic projections we obtain a map

$$S^{q+r} \longrightarrow S^{2q+r}$$

between spheres. Any such map is nullhomotopic [4, Proposition 17.9]. Therefore, the induced K-theory map

$$K^j(S^{2q+r}) \longrightarrow K^j(S^{q+r})$$

is the zero map.

The Case $n = 2$

For $n = 2$ there are two Levi subgroups of $GL(2, \mathbb{R})$, $M_1 \cong GL(2, \mathbb{R})$ and the diagonal subgroup $M_2 \cong (\mathbb{R}^\times)^2$. By Proposition 2.2 $X(M_1)$ is parametrized by $\mathbb{R}$ and $X(M_2)$ is parametrized by $\mathbb{R}^2$. The group $GL(2, \mathbb{C})$ has only one Levi subgroup, the diagonal subgroup $M \cong (\mathbb{C}^\times)^2$. From Proposition 2.3 it is parametrized by $\mathbb{R}^2$.

Since $K^1(\mathcal{A}_2^t(\mathbb{C})) = 0$ by Theorem 5.1, we only have to consider the K^0 functor. The only contribution to $K^0(\mathcal{A}_2^t(\mathbb{R}))$ comes from $M_2 \cong (\mathbb{R}^\times)^2$ and we have (see Example 4.5)

$$K^0(\mathcal{A}_2^t(\mathbb{R})) \cong \mathbb{Z}.$$

For the Levi subgroup $M_2 \cong (\mathbb{R}^\times)^2$, base change is

$$\begin{array}{ccc} BC : \mathcal{A}_2^t(\mathbb{R}) & \longrightarrow & \mathcal{A}_2^t(\mathbb{C}) \\ \pi(\xi_1, \xi_2) & \mapsto & i_{GL(2,\mathbb{C}),B(\mathbb{C})}(\xi_1 \circ N, \xi_2 \circ N), \end{array}$$

Therefore, it maps a class $[t_1, t_2]$, which lies in the connected component $(\varepsilon_1, \varepsilon_2)$, into the class $[2t_1, 2t_2]$, which lies in the connect component $(0, 0)$. In other words, base change maps a generalized principal series $\pi(\xi_1, \xi_2)$ into a nongeneric point of $\mathcal{A}_2^t(\mathbb{C})$. It follows from Theorem 5.1 that

$$K^0(BC) : K^0(\mathcal{A}_2^t(\mathbb{R})) \to K^0(\mathcal{A}_2^t(\mathbb{C}))$$

is the zero map. $\square$

7 Base change in one dimension

In this section we consider base change for $GL(1)$.

Theorem 7.1. *The functorial map induced by base change*

$$K_1(C_r^*GL(1,\mathbb{C})) \xrightarrow{K_1(BC)} K_1(C_r^*GL(1,\mathbb{R}))$$

is given by $K_1(BC) = \Delta \circ Pr$, *where* Pr *is the projection of the zero component of* $K^1(\mathcal{A}_1^t(\mathbb{C}))$ *into* $\mathbb{Z}$ *and* Δ *is the diagonal* $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$.

Proof. For $GL(1)$, base change

$$\chi \in \mathcal{A}_1^t(\mathbb{R}) \mapsto \chi \circ N_{\mathbb{C}/\mathbb{R}} \in \mathcal{A}_1^t(\mathbb{C})$$

induces a map

$$K^1(BC) : K^1(\mathcal{A}_1^t(\mathbb{C})) \to K^1(\mathcal{A}_1^t(\mathbb{R})).$$

Any character $\chi \in \mathcal{A}_1^t(\mathbb{R})$ is uniquely determined by a pair of parameters $(t, \varepsilon) \in \mathbb{R} \times \mathbb{Z}/2\mathbb{Z}$. Similarly, any character $\mu \in \mathcal{A}_1^t(\mathbb{C})$ is uniquely determined by a pair of parameters $(t, \ell) \in \mathbb{R} \times \mathbb{Z}$. The discrete parameter ε (resp., ℓ) labels each connected component of $\mathcal{A}_1^t(\mathbb{R}) = \mathbb{R} \sqcup \mathbb{R}$ (resp., $\mathcal{A}_1^t(\mathbb{C}) = \bigsqcup_{\mathbb{Z}} \mathbb{R}$).

Base change maps each component ε of $\mathcal{A}_1^t(\mathbb{R})$ into the component 0 of $\mathcal{A}_1^t(\mathbb{C})$, sending $t \in \mathbb{R}$ to $2t \in \mathbb{R}$. The map $t \mapsto 2t$ is homotopic to the identity. At the level of K^1, the base change map may be described by the following commutative diagram

$$\begin{tikzcd} \bigoplus_{\widehat{\mathbb{T}}} \mathbb{Z} \arrow[r, "K^1(BC)"] \arrow[d, "Pr"'] & \mathbb{Z} \oplus \mathbb{Z} \\ \mathbb{Z} \arrow[ur, "\Delta"'] & \end{tikzcd}$$

Here, Pr is the projection of the zero component of $K^1(\mathcal{A}_1^t(\mathbb{C})) \cong \bigoplus_{\widehat{\mathbb{T}}} \mathbb{Z}$ into $\mathbb{Z}$ and Δ is the diagonal map. $\square$

References

[1] J. Arthur, L. Clozel, Simple algebras, base change, and the advanced theory of the trace formula, Annals of Math. Studies 120, Princeton University Press, Princeton, 1989.

[2] P. Baum, A. Connes, N. Higson, Classifying space for proper actions and K-theory for group C^*-algebras, Contemporary Math., 167 (1994) 241–291.

[3] J. Bernstein, S. Gelbart et al, An introduction to the Langlands program, Birkhauser 2003.

[4] R. Bott and L.W. Tu, Differential Forms in Algebraic Topology, Springer-Verlag, New York 1982.

[5] J. Chabert, S. Echterhoff, R. Nest, The Connes-Kasparov conjecture for almost connected groups and for linear p-adic groups, Publications Math. I.H.E.S. 97 (2003) 239-278.

[6] Harish-Chandra, Collected papers, Vol. 4, Springer, Berlin (1984).

[7] A. W. Knapp, Local Langlands correspondence: the archimedean case, Proc. Symp. Pure. Math. 55, 1994, 393–410.

[8] V. Lafforgue, K-théorie bivariante pour les algèbres de Banach et conjecture de Baum-Connes, Invent. Math. 149 (2002) 1–95.

[9] S. Mendes, R.J. Plymen, Base change and K-theory for $GL(n)$, J. Noncommut. Geom., 1 (2007), 311–331.

[10] R. J. Plymen, The reduced C*-algebra of the p-adic group $GL(n)$, J. Functional Analysis 72 (1987) 1–12.

[11] M. G. Penington, R. J. Plymen, The Dirac operator and the principal series for complex semisimple Lie groups, J. Functional Analysis, 53 (1983) 269–286.

[12] M. Rørdam, F. Larsen, N. J. Lausten, An Introduction to K-Theory for C^*-Algebras, London Math Soc. Student Texts 49, Cambridge University Press, Cambridge, 2000.

[13] J. Tate, Number theoretic background, Proc. Symp. Pure Math. 33 (1979) part 2, 3–26.

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